Combining Runge-Kutta discontinuous Galerkin methods with various limiting methods

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Overview

- Discontinuous Galerkin Finite Element Methods (DG-FEM)
  - Constructing a DG-FEM scheme for the linear advection equation
  - Example results
- Time discretisation
  - Explicit TVD-Runge-Kutta (TVD-ERK) schemes
  - Implicit time discretisation
  - Implicit Runge-Kutta (IRK) schemes
- Limiting-Methods
  - Slope-Limiter
  - Moment-Limiter
  - Limiting IRK schemes

DG-FEM Scheme

How to construct a discontinuous Galerkin finite element (DG-FEM) scheme?

Linear Advection Equation

$$\frac{\partial}{\partial t} f(x,t) + a \frac{\partial}{\partial x} f(x,t) = 0 \tag{1}$$

Linear Advection Equation

$$\frac{\partial}{\partial t} f(x,t) + a \frac{\partial}{\partial x} f(x,t) = 0 \quad (1)$$

Residual

$$\frac{\partial f_i}{\partial t} + a \frac{\partial f_i}{\partial x} = \xi_i(x,t) \quad (2)$$

Linear Advection Equation

$$\frac{\partial}{\partial t} f(x,t) + a \frac{\partial}{\partial x} f(x,t) = 0 \tag{1}$$

orthogonal test functions $\nu(x)$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \nu \left(\frac{\partial f_i}{\partial t} + a \frac{\partial f_i}{\partial x} \right) \, \mathrm{d}x = 0 \tag{2}$$

Linear Advection Equation

$$\frac{\partial}{\partial t} f(x,t) + a \frac{\partial}{\partial x} f(x,t) = 0 \tag{1}$$

integration by parts

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\nu(x) \frac{\partial f_i(x)}{\partial t} - a \frac{\partial \nu(x)}{\partial x} f_i(x) \right) \mathrm{d}x = - \Big[\nu(x) a f_i(x) \Big]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \tag{2}$$

Linear Advection Equation

$$\frac{\partial}{\partial t} f(x,t) + a \frac{\partial}{\partial x} f(x,t) = 0 \qquad (1)$$

integration by parts

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\nu(x) \frac{\partial f_i(x)}{\partial t} - a \frac{\partial \nu(x)}{\partial x} f_i(x) \right) \, \mathrm{d}x = - \Big[\nu(x) a f_i(x) \Big]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \qquad (2)$$

Galerkin's Approach

test and base functions from same class

$$\nu(x), f_i(x) \in V^p(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$$

Linear

(1)

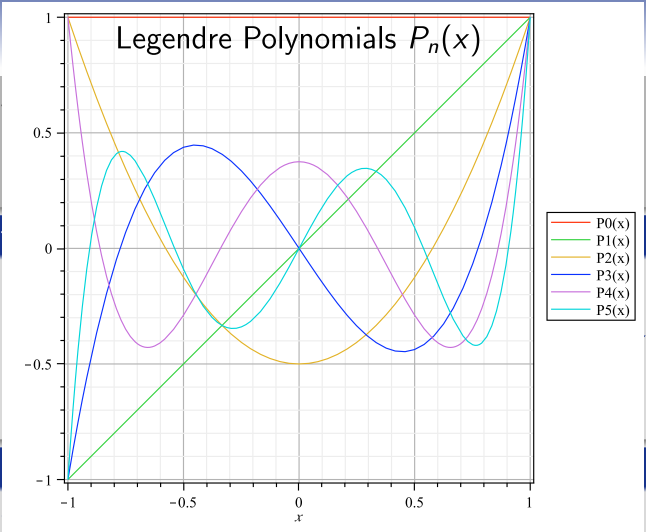
integra

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \quad \Big]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \tag{2}$$

Galerki

test an

$$\nu(x), f_i(x) \in V^p(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$$



Integrated Residual

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\nu(x) \frac{\partial f_i(x)}{\partial t} - a \frac{\partial \nu(x)}{\partial x} f_i(x) \right) \, \mathrm{d}x = - \Big[\nu(x) a f_i(x) \Big]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \tag{3}$$

Use Polynomials

$$\begin{aligned} & \sum_{m=0}^{p} \left\{ \left(\int_{-1}^{+1} P_n(\tilde{x}) P_m(\tilde{x}) \frac{\mathrm{d}x}{\mathrm{d}\tilde{x}} \, \mathrm{d}\tilde{x} \right) \frac{\mathrm{d}\vec{u}_{i,m}}{\mathrm{d}t} \right\} \\ + \quad & a \sum_{m=0}^{p} \left\{ \left(\int_{-1}^{+1} P_n(\tilde{x}) \frac{\mathrm{d}}{\mathrm{d}x} P_m(\tilde{x}) \frac{\mathrm{d}x}{\mathrm{d}\tilde{x}} \, \mathrm{d}\tilde{x} \right) \vec{u}_{i,m} \right\} \\ = \quad & a P_n(-1) \sum_{m=0}^{p} P_m(1) u_{i-1,m} - a P_n(-1) \sum_{m=0}^{p} P_m(-1) u_{i,m} \end{aligned} \tag{4}$$

Use Polynomials

$$
\begin{aligned}
& \sum_{m=0}^{p} \left\{ \left(\int_{-1}^{+1} P_n(\tilde{x}) P_m(\tilde{x}) \frac{\mathrm{d}x}{\mathrm{d}\tilde{x}} \, \mathrm{d}\tilde{x} \right) \frac{\mathrm{d}\vec{u}_{i,m}}{\mathrm{d}t} \right\} \\
+ \quad & a \sum_{m=0}^{p} \left\{ \left(\int_{-1}^{+1} P_n(\tilde{x}) \frac{\mathrm{d}}{\mathrm{d}x} P_m(\tilde{x}) \frac{\mathrm{d}x}{\mathrm{d}\tilde{x}} \, \mathrm{d}\tilde{x} \right) \vec{u}_{i,m} \right\} \\
= \quad & a P_n(-1) \sum_{m=0}^{p} P_m(1) u_{i-1,m} - a P_n(-1) \sum_{m=0}^{p} P_m(-1) u_{i,m}
\end{aligned}
\tag{3}
$$

Matrices of DG-FEM

$$
\mathcal{M}_{nm} := \frac{\mathrm{d}x}{\mathrm{d}\tilde{x}} \int_{-1}^{+1} P_n(\tilde{x}) P_m(\tilde{x}) \, \mathrm{d}\tilde{x} \tag{4}
$$

$$
\mathcal{D}_{nm} := \frac{\mathrm{d}x}{\mathrm{d}\tilde{x}} \int_{-1}^{+1} P_n(\tilde{x}) \frac{\mathrm{d}}{\mathrm{d}x} P_m(\tilde{x}) \, \mathrm{d}\tilde{x} \tag{5}
$$

Use Polynomials

$$
\begin{aligned}
& \sum_{m=0}^{p} \left\{ \left(\int_{-1}^{+1} P_n(\tilde{x}) P_m(\tilde{x}) \frac{\mathrm{d}x}{\mathrm{d}\tilde{x}} \,\mathrm{d}\tilde{x} \right) \frac{\mathrm{d}\vec{u}_{i,m}}{\mathrm{d}t} \right\} \\
+ \quad & a \sum_{m=0}^{p} \left\{ \left(\int_{-1}^{+1} P_n(\tilde{x}) \frac{\mathrm{d}}{\mathrm{d}x} P_m(\tilde{x}) \frac{\mathrm{d}x}{\mathrm{d}\tilde{x}} \,\mathrm{d}\tilde{x} \right) \vec{u}_{i,m} \right\} \qquad (3) \\
= \quad & a P_n(-1) \sum_{m=0}^{p} P_m(1) u_{i-1,m} - a P_n(-1) \sum_{m=0}^{p} P_m(-1) u_{i,m}
\end{aligned}
$$

Matrices of DG-FEM

$$
\begin{aligned}
\mathcal{F}^{+}_{nm} &:= P_n(-1) P_m(1) \qquad (4) \\
\mathcal{F}^{-}_{nm} &:= P_n(-1) P_m(-1) \qquad (5)
\end{aligned}
$$

Matrices of DG-FEM

$$\mathcal{M}_{nm} := \frac{\mathrm{d}x}{\mathrm{d}\tilde{x}} \int_{-1}^{+1} P_n(\tilde{x}) P_m(\tilde{x})\, \mathrm{d}\tilde{x} \tag{6}$$

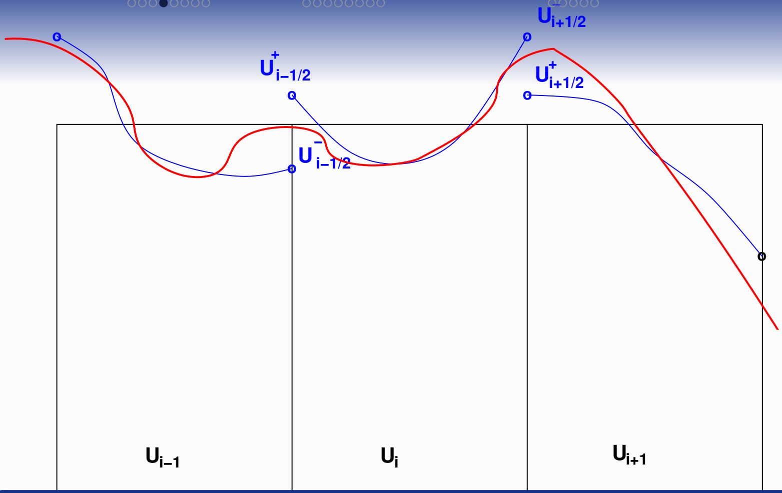
$$\mathcal{D}_{nm} := \frac{\mathrm{d}x}{\mathrm{d}\tilde{x}} \int_{-1}^{+1} P_n(\tilde{x}) \frac{\mathrm{d}}{\mathrm{d}x} P_m(\tilde{x})\, \mathrm{d}\tilde{x} \tag{7}$$

$$\mathcal{F}^{+}_{nm} := P_n(-1) P_m(1) \tag{8}$$

$$\mathcal{F}^{-}_{nm} := P_n(-1) P_m(-1) \tag{9}$$

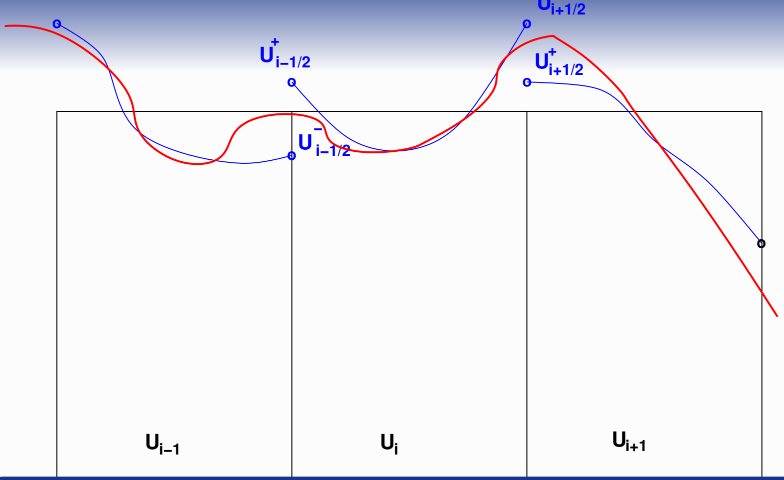
Linear Advection Equation discretised with DG-FEM

$$\mathcal{M} \frac{\mathrm{d}\vec{u}_i}{\mathrm{d}t} = -a \left[\mathcal{D} \vec{u}_i - \left(\mathcal{F}^{+} \vec{u}_{i-1} - \mathcal{F}^{-} \vec{u}_i \right) \right] \tag{10}$$



Linear Advection Equation

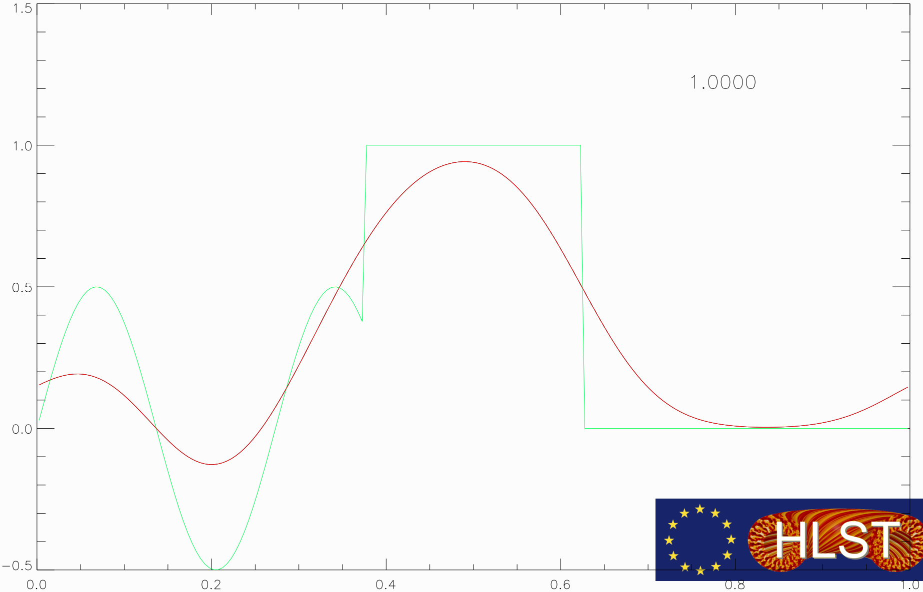
$$\frac{\partial}{\partial t} f(x,t) + a \frac{\partial}{\partial x} f(x,t) = 0 \quad (11)$$



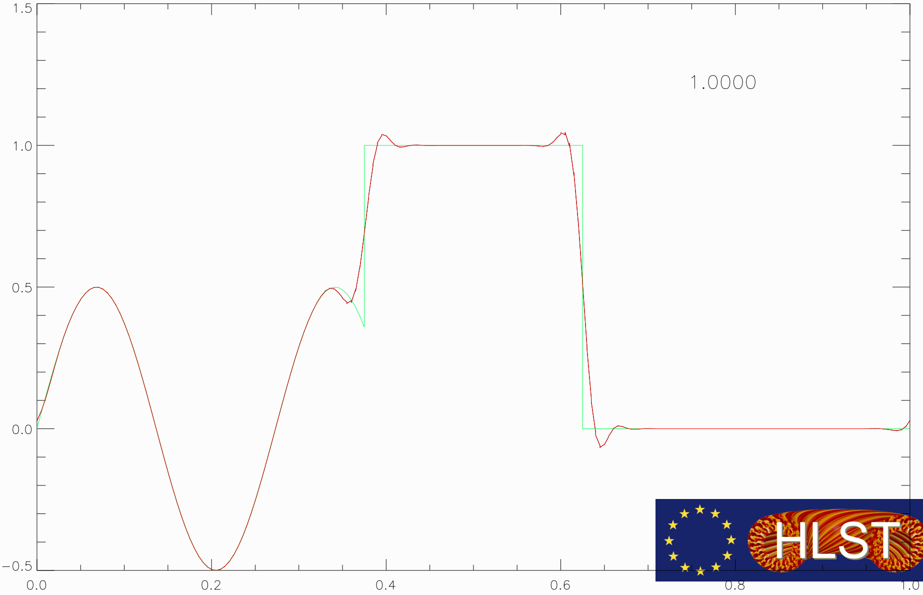
Linear Advection Equation discretised with DG-FEM

$$\mathcal{M}\frac{\mathrm{d}\vec{u}_i}{\mathrm{d}t} = -a\left[\mathcal{D}\vec{u}_i - \left(\mathcal{F}^{+}\vec{u}_{i-1} - \mathcal{F}^{-}\vec{u}_i\right)\right] \tag{11}$$

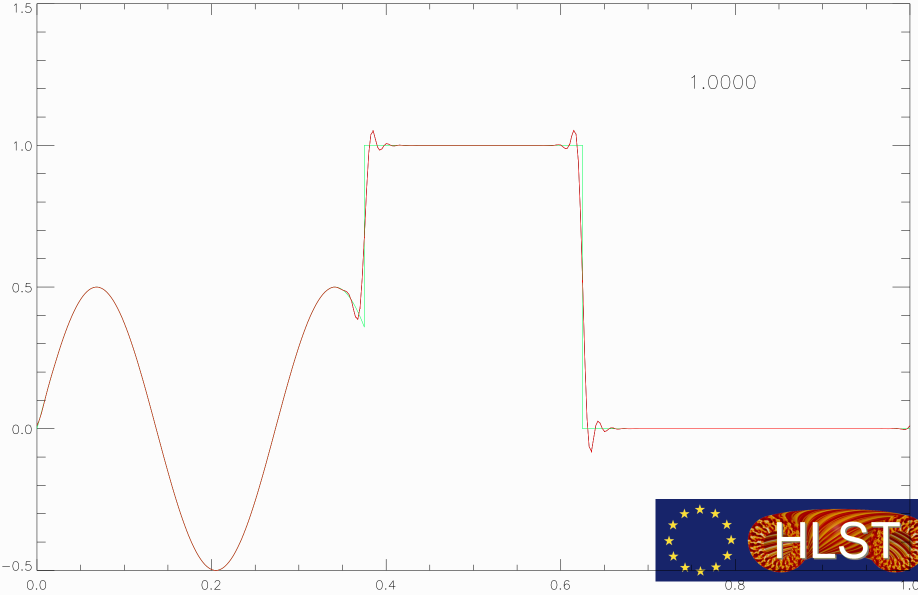
DG-FEM: $p = 0$, cfl $= 0.1$



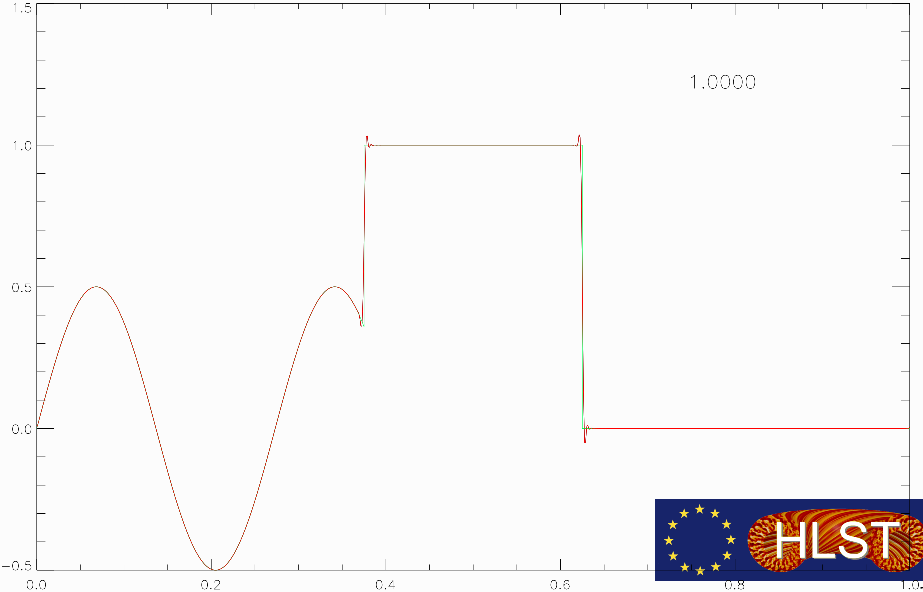
DG-FEM: $p = 1$, $cfl = 0.1$



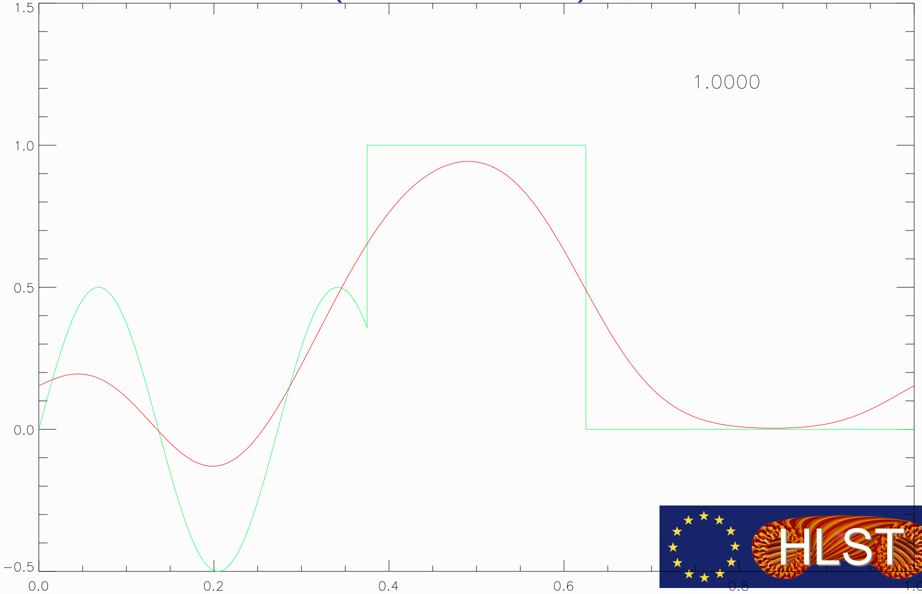
DG-FEM: p = 2, cfl = 0.1



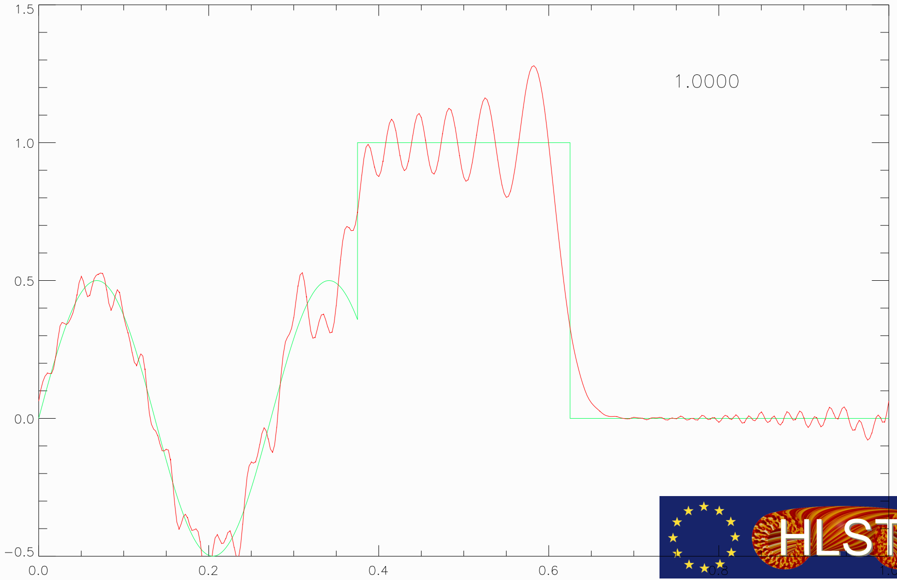
DG-FEM: $p = 6$, $\mathrm{cfl} = 0.05$



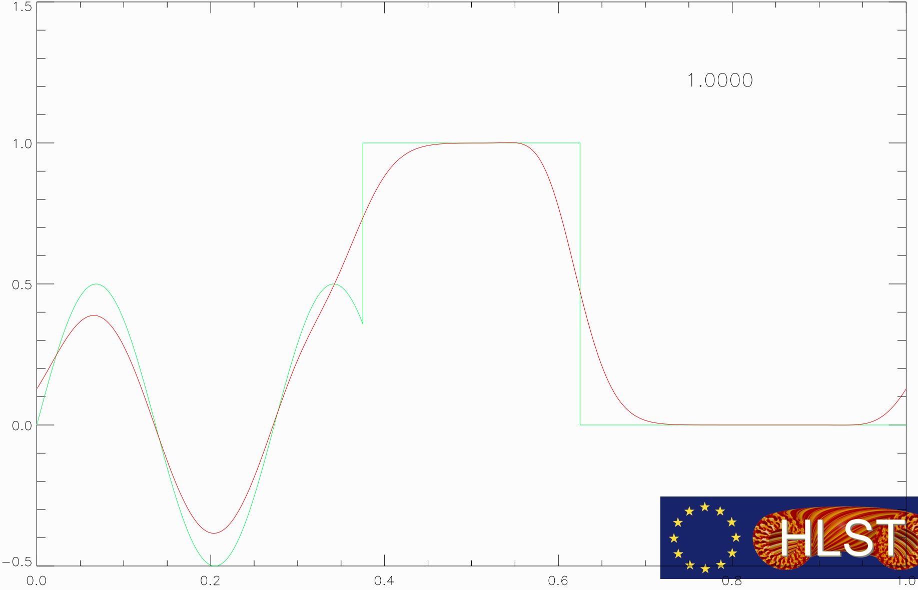
impl. DG-FEM (Euler backw.): $p = 2$, cfl $= 1$.



impl. DG-FEM (Crank-Nichols.): $p = 2$, cfl $= 1.0$



impl. DG-FEM (gen. CN): $p = 2$, cfl $= 1.0$



General Runge-Kutta Equation System

$$\vec{u}^{(1)} = f\left(t^n + c_1\Delta t, \vec{u}^{\,n} + \sum_{k=1}^{s} a_{1k}\Delta t \vec{u}^{\,(k)}\right) \tag{12}$$

$$\vec{u}^{(2)} = f\left(t^n + c_2\Delta t, \vec{u}^{\,n} + \sum_{k=1}^{s} a_{2k}\Delta t \vec{u}^{\,(k)}\right) \tag{13}$$

$$\vdots$$

$$\vec{u}^{(s)} = f\left(t^n + c_s\Delta t, \vec{u}^{\,n} + \sum_{k=1}^{s} a_{sk}\Delta t \vec{u}^{\,(k)}\right) \tag{14}$$

$$\vec{u}^{\,n+1} = f\left(t^n + \Delta t, \vec{u}^{\,n} + \sum_{k=1}^{s} a_{(s+1)k}\Delta t \vec{u}^{\,(k)}\right) \tag{15}$$

General Runge-Kutta Equation System

$$\vec{u}^{(1)} = f\left(t^n + c_1\Delta t, \vec{u}^{\,n} + \sum_{k=1}^{s} a_{1k}\Delta t \vec{u}^{(k)}\right) \qquad (12)$$

$$\vec{u}^{(2)} = f\left(t^n + c_2\Delta t, \vec{u}^{\,n} + \sum_{k=1}^{s} a_{2k}\Delta t \vec{u}^{(k)}\right) \qquad (13)$$

$$\vdots$$

Butcher's Array

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & a_{t1} & a_{t2} & \cdots & a_{ts} \end{array} \quad . \qquad (16)$$

Butcher's Array

$$
\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\
c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\
\hline
 & a_{t1} & a_{t2} & \cdots & a_{ts}
\end{array}
\tag{17}
$$

3rd Order TVD-ERK (3 stages)

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
\frac{1}{4} & \frac{3}{4} & \frac{1}{4} & 0 \\
\hline
 & \frac{1}{3} & \frac{2}{3} & 0
\end{array}
\tag{18}
$$

Butcher's Array

$$
\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\
c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\
\hline
 & a_{t1} & a_{t2} & \cdots & a_{ts}
\end{array}
\tag{17}
$$

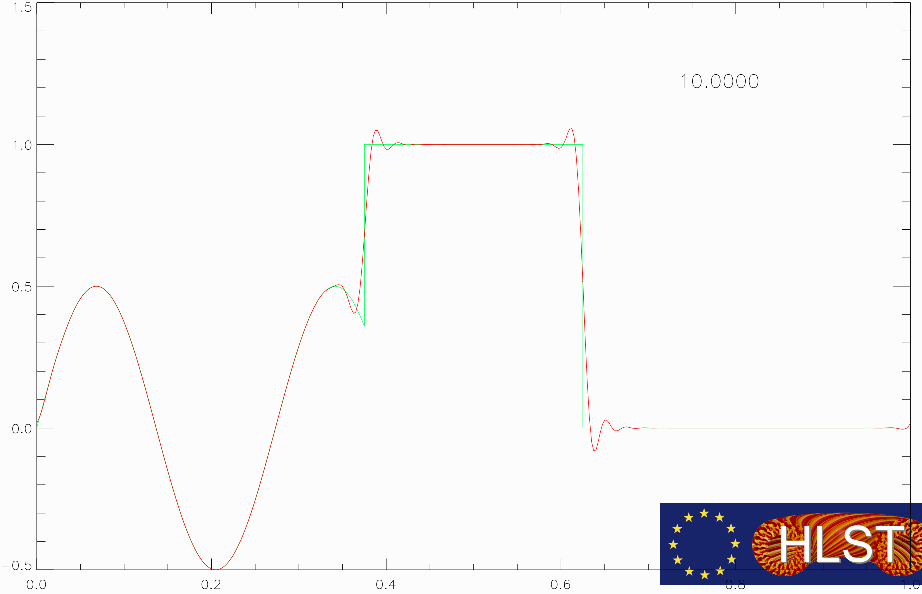
3rd Order TVD-ERK (3 stages)

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0
\end{array}
$$

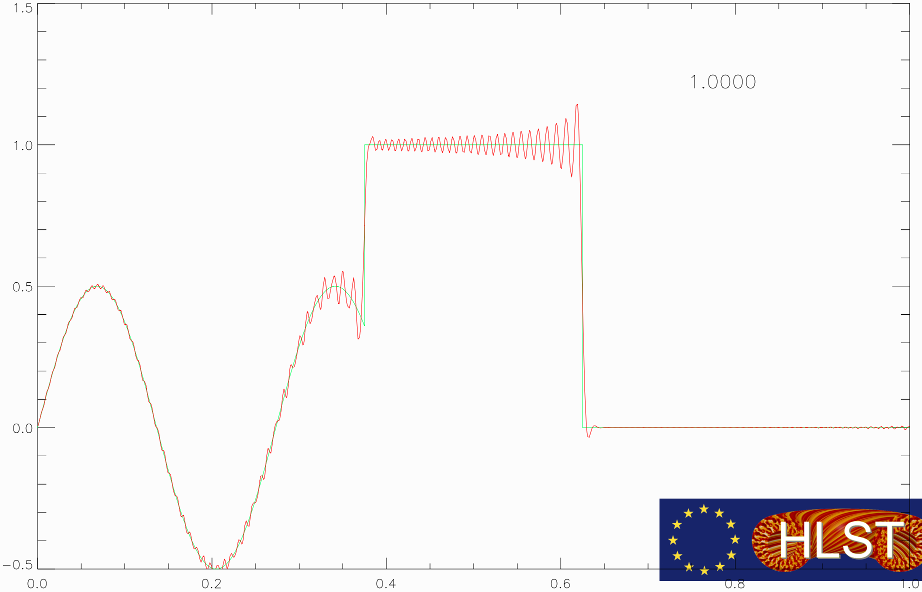
6th Order GL-IRK (3 stages)

$$
\begin{array}{c|ccc}
\frac{1}{2} - \frac{\sqrt{15}}{10} & \frac{5}{36} & \frac{2}{9} - \frac{\sqrt{15}}{15} & \frac{5}{36} - \frac{\sqrt{15}}{30} \\
\frac{1}{2} & \frac{5}{36} + \frac{\sqrt{15}}{24} & \frac{2}{9} & \frac{5}{36} - \frac{\sqrt{15}}{24} \\
\frac{1}{2} + \frac{\sqrt{15}}{10} & \frac{5}{36} + \frac{\sqrt{15}}{30} & \frac{2}{9} + \frac{\sqrt{15}}{15} & \frac{5}{36} \\
\hline
 & \frac{5}{18} & \frac{4}{9} & \frac{5}{18}
\end{array}
\tag{19}
$$

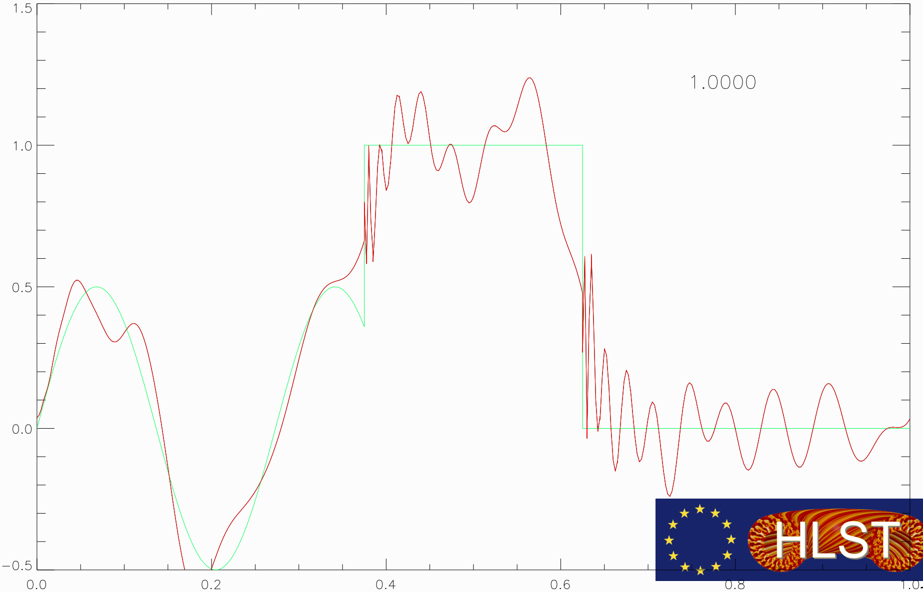
impl. RK-DG-FEM (SDIRK4th): $p = 2$, cfl $= .5$



IRK (GL 8th) DG-FEM : $p = 4$, cfl $= 1.6$



IRK (GL 8th) DG-FEM : $p = 2$, cfl $= 25.0$



Limiting Methods

Basic Idea

Wiggles form in the higher order moments at steep gradients
$\Longrightarrow$ limit higher order moments by lower order moments

Limiting Methods

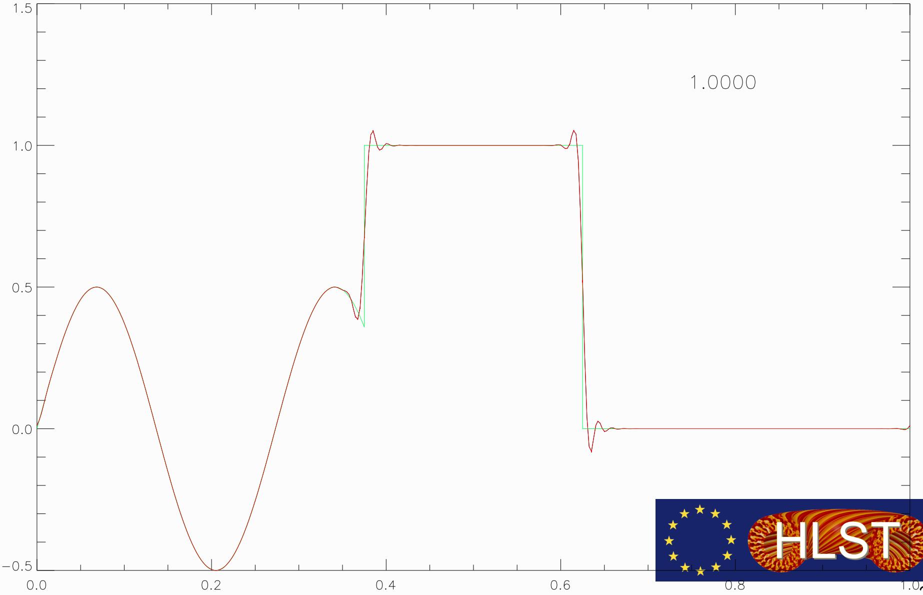
Basic Idea

Wiggles form in the higher order moments at steep gradients
$\Longrightarrow$ limit higher order moments by lower order moments

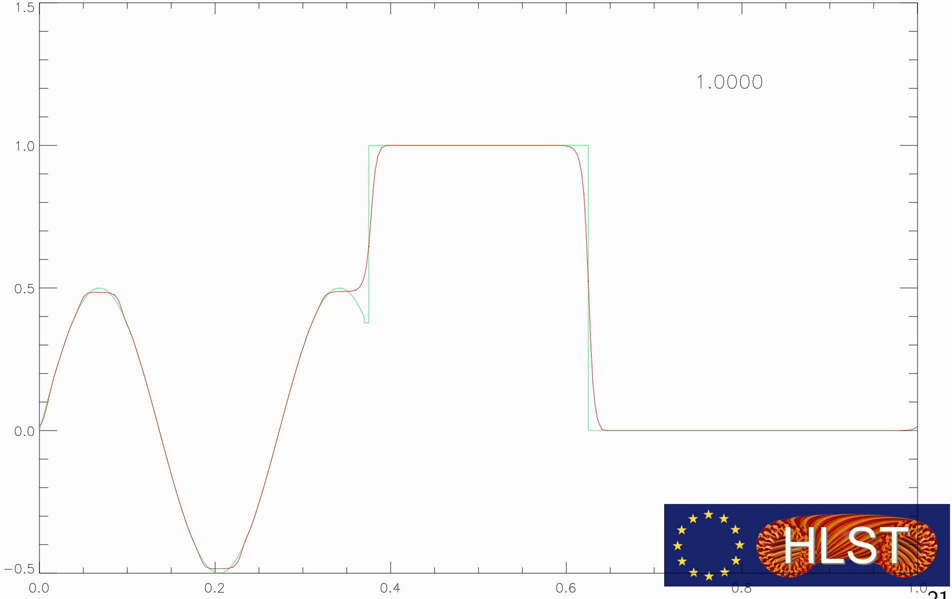
Moment Limiter as Example

$$u_i^{n+1} = \mathrm{minmod}\left(u_i^{n+1}, \frac{u_{i+1}^n - u_i^n}{2n+1}, \frac{u_i^n - u_{i-1}^n}{2n+1}\right) \tag{20}$$

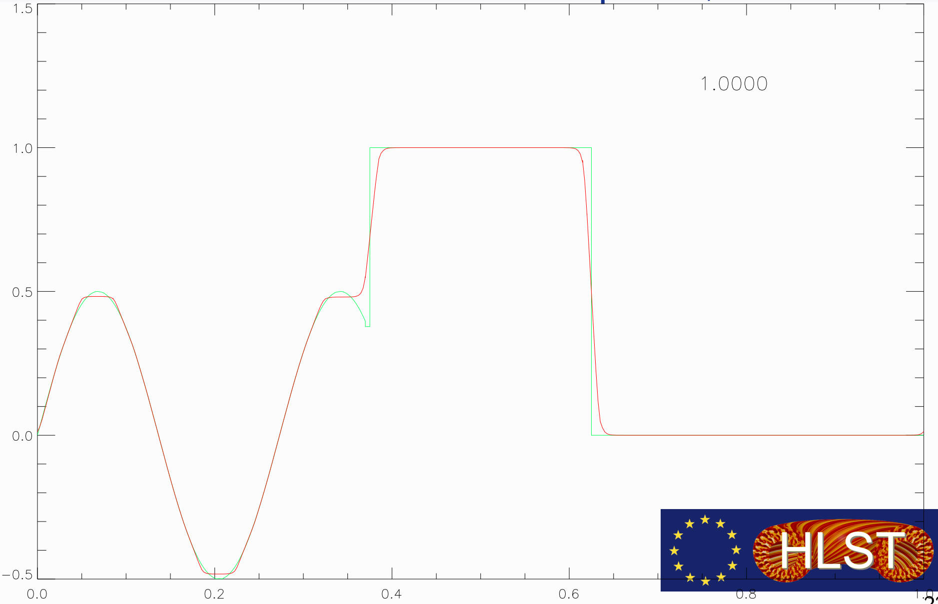
DG-FEM: p = 2, cfl = 0.1



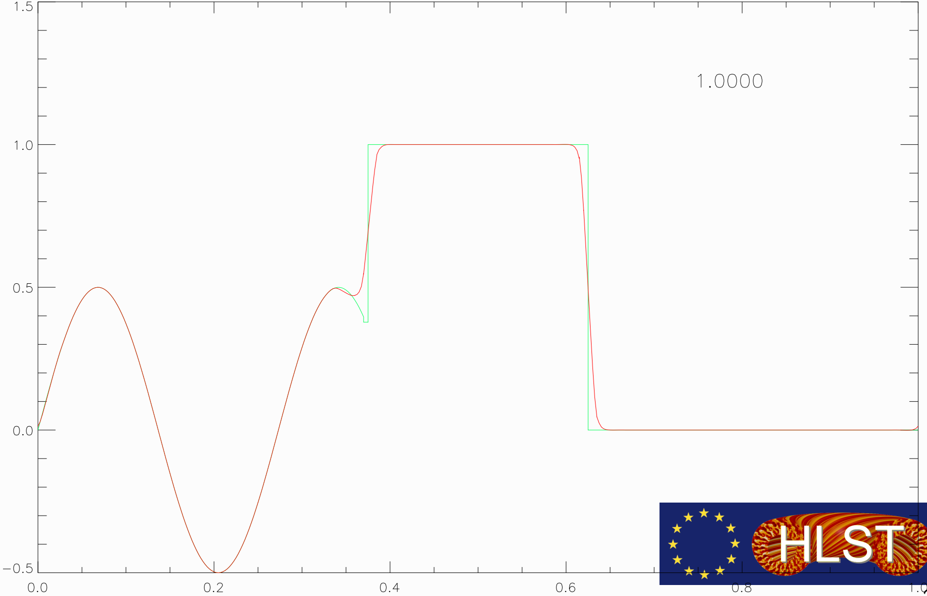
DG-FEM + Slope-Limiter: p = 2, cfl = 0.1



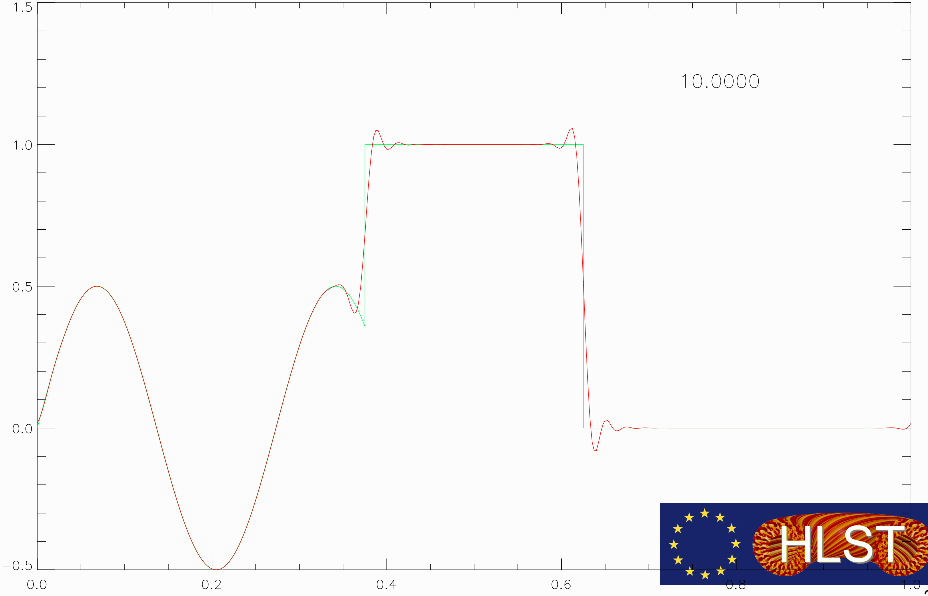
DG-FEM + Moment-Limiter: p = 2, cfl = 0.1



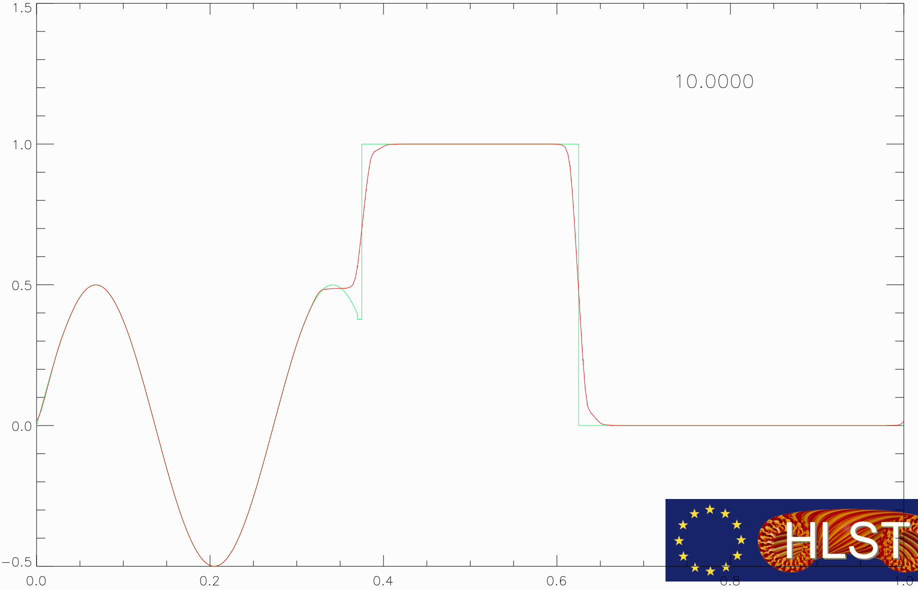
DG-FEM + gen. Moment-Limiter: $p = 2$, cfl $= 0.1$



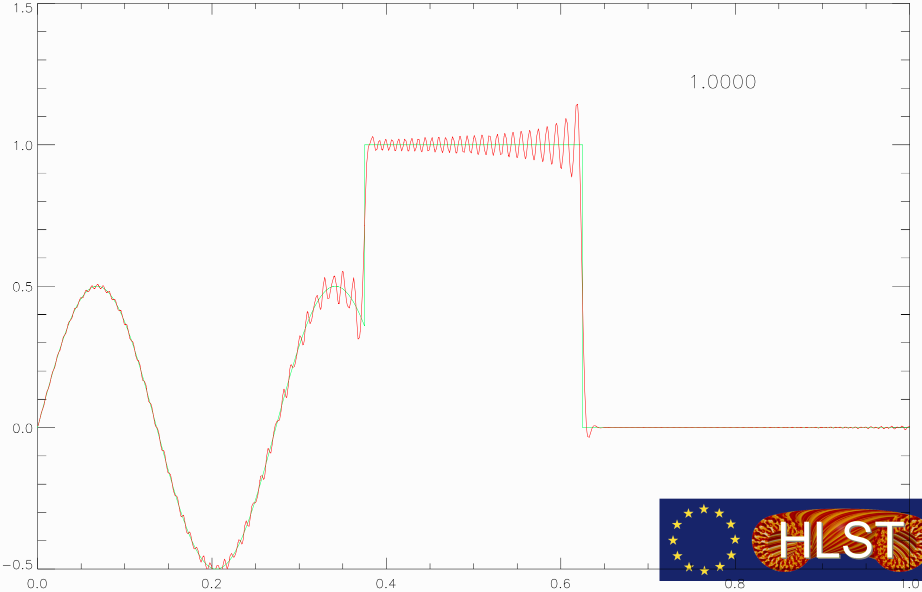
impl. RK-DG-FEM (SDIRK4th): p = 2, cfl = .5



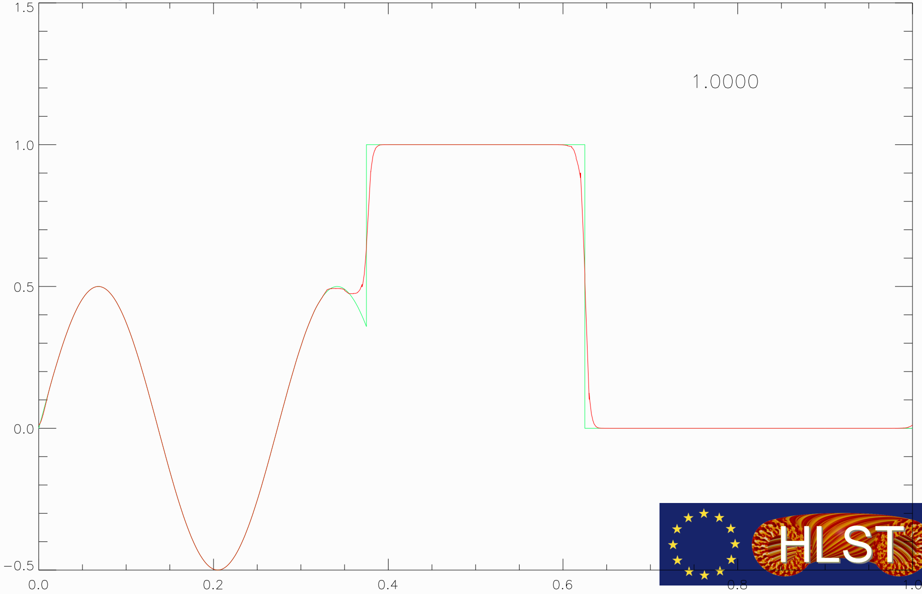
SDIRK-DG-FEM (4th) + GML: $p = 2$, cfl $= .5$



IRK (GL 8th) DG-FEM : p = 4, cfl = 1.6



IRK (GL 8th) DG-FEM + GML: $p = 4$, cfl $= 1.6$



More Details

More details can be found in the technical report

Combining Runge-Kutta discontinuous Galerkin methods with various limiting methods

`http://edoc.mpg.de/display.epl?mode=doc&id=446381`